Containing and managing an emerging disease outbreak: a stochastic modelling approach.

Idriss SEKKAK[1,3], Mohamed El FATINI[1], and Jude Dzevela KONG[2]

[1]Laboratory of PDE's, Algebra and Spectral Geometry. Faculty of Sciences, Ibn Tofail University, BP 133,Kénitra, Morocco
[2]Africa-Canada Artificial Intelligence and Data Innovation Consortium (ACADIC)
[3]Mathematics for Public Health, The Fields Institute for Research in Mathematical Sciences, 222 College Street · Toronto, Ontario · M5T 3J1 · Canada

February 17, 2022

Abstract

The aim of this work is to design and analyze a novel stochastic model for an infectious disease transmission dynamics, that captures human responses to information about the disease, policy, and disease progression in the event of an outbreak. We design a behaviour-structured stochastic Susceptible-Infected-Quarantine-Recovered model incorporating a population logistic growth, non pharmaceutical interventions and a general functional response in order to capture respectively the long time growth of a population size, instant measures established by decision makers and human response behaviour. We carry out a thorough analysis to investigate the existence of the global and positive solutions and to explore the extinction and the persistence of the disease regarding the basic reproduction number of the model. Moreover, we use suitable Lyapunov functions and establish sufficient conditions for the existence of ergodic stationary distribution of the solution to the stochastic SIQR model. In addition, we estimate the parameters of the model by fitting it to confirmed COVID19 cases in Morocco using least squares method.

Keywords– Stochastic process, Epidemic model, Extinction, Stationary distribution, Data fitting.

1 Introduction

Over time, a population fluctuates in composition and size due to multiple factors, such as immigration, diseases, natural birth and death, etc. Therefore, a variety of mathematical models tend to explore population dynamics as a way to describe the occurring changes in order to predict its behaviour. However, a sudden rapid increase in the number of cases of a new emerging infectious disease requires an instant response in order to lighten the burden on health care institutions. Currently, during the COVID-19 outbreak, the use of compartmental models, derived from the classical SIR (Susceptible-Infected-Recovered) epidemic model proposed by Kermack and McKendrick [1], provided an insight on how to react to an emerging infectious

disease. Many works [2, 3, 4, 5, 6] are developed to reflect the epidemiological and environmental characteristics of the dynamical behaviour in a population. Several of the epidemic models illustrated development without regard for the environment, while others works [7, 8, 9, 10] included a ceiling imposed by resource constraints, where a population's size reaches a maximum dictated by restricted resources in the system, defined as the carrying capacity K, the population's per capita growth rate decreases. Yet, when an emerging infectious disease last for a long time without a lasting prophylaxis such as a efficient vaccine, we consider a long time behaviour for the population growth instead of a fixed recruitment rate into the population. Therefore, the only efficient and instant measures left for a policy maker are the Non-pharmaceutical interventions.

1.1 Non-pharmaceutical interventions

As a control strategy, non pharmaceutical Interventions (NPIs) [11, 12, 13] are measures that individuals and groups should be doing in addition to being vaccinated and receiving treatment that further reduce the transmission of illnesses like COVID-19, Ebola and influenza. Those measures are also known as community mitigation policies. Mainly, they are used when a pandemic virus is new, and a worldwide human population has a weak or no immunity against. In [11], authors analyzed the impact of other factors other than NPI, where they explored the effect demographic, social and climatic variables on the spread of a COVID-19 disease. On the other hand, an epidemic results to some psychological effects on the individuals and their behavior and a saturation level due to the increase of the infected population resulting to a decrease of the infection force. Therefore, Capasso and Serio [14] proposed a generalized functional response to describe the impact of those effects.

1.2 The loss of immunity

The existence of antibodies to a disease in a person's system offers immunity against that infection. Antibodies are proteins that the immune system produces to neutralize or eliminate pathogens, which have a disease-specific function. If an immune individual comes into touch with an infectious disease that have been recognized, their immune system will identify it and generate the antibodies necessary to tackle it right away. In [15], Corona viruses antibodies, as an example, will protect a person if he or she is exposed to the infection leading to lungs complications but will probably have no impact after several months of exposure.

The paper is organized as follows. In section methods (2), we present a deterministic and a stochastic formulation of the epidemic model, we explore the non pharmaceutical interventions established by the Moroccan government and the collected data for the calibration with our model. In section results (3), we show the existence of the global and positive solutions and their boundedness. We establish sufficient conditions for the extinction of the disease. We obtain sufficient conditions for positive recurrence and ergodic properties of system (3) in order to show that the disease may persist in a population. We fit a dataset of confirmed daily COVID19 cases in Morocco into the epidemic model (1) using least squares method in order to estimate the unknown parameters and we provide a discussion on the numerical illustrations. In section (4), we give some concluding remarks and our future perspectives.

2 Methods

2.1 Model formulation

These epidemic modelling challenges are a helpful tool for a government to adopt a public health strategy or measure. For that reason, many mathematical epidemic included a quarantine policy in order to predict the development trend; since it costs a lot on the economic level. Thus, in [16, 17] authors studied the impact of random perturbations of an SIQR epidemic model. Also, Liu et al [18] explored a multi group SIQR model; and authors, in [5, 6], included the impact of delay to describe an immunity loss. For this matter, we investigate in this work a SIQRS epidemic with logistic growth, non pharmaceutical interventions and a generalized functional response:

$$\begin{aligned} dS(t) &= \left[rS(t)\left(1-\frac{S(t)}{K}\right) - \beta M(t)S(t)f(I) + \delta R(t)\right]dt, \\ dI(t) &= \left[\beta M(t)S(t)f(I) - (\mu+\alpha_1+\lambda+\rho)I(t)\right]dt, \\ dQ(t) &= \left[\lambda I(t) - (\mu+\alpha_2+\varepsilon)Q(t)\right]dt \\ dR(t) &= \left[\rho I(t) + \varepsilon Q(t) - (\mu+\delta)R(t)\right]dt, \end{aligned} \tag{1}$$

where $S(t)$ denotes the susceptible individuals, $I(t)$ for infected compartment, $Q(t)$ for quarantined or isolated individuals and $R(t)$ for recovered compartment. In addition to an intrinsic growth rate r, the susceptible population are maintained by logistic growth including a carrying capacity K. The parameter μ denotes the natural death rate of S, I, Q and R compartments, β denotes the transmission coefficient from susceptible to infected individuals, ρ describes the recovery rate of the infective individuals, α_1 and α_2 represents the death rate for infected and quarantined individuals because of infection complications, ρ denotes the rate of infectious individuals who were isolated, ε represents the recovered people coming from isolation, λ describes infected people who recovered from the infection, and δ denotes individuals that lost the immunity to the infectious disease. In addition, The general incidence function $f(.)$ [19, 20, 21, 22] is nonnegative and bounded, twice continuously differentiable and $\frac{f(.)}{.}$ is monotonically decreasing on $[0,\infty)$, with $f(0) = 0$ and $f'(0) > 0$. Hence, $\frac{f(x)}{x} \leq f'(0)$ for any $x > 0$. The function $M(t)$ is a predetermined function [12] that captures the effects of non-pharmaceutical interventions on rates of infection. When fitting, we use a functional form of

$$M(t) = a + (1-a)e^{-mt}. \tag{2}$$

Besides, inspired by environmental facts, a perturbation of a certain parameter in an epidemic model can describe factors such as secondary diseases resulting to complications, air pollution, climate change, etc, where in case of unknown parameters affecting the system, it becomes subject to stochastic disturbances. Hence, the epidemic model (1) is described as follows

$$\begin{aligned} dS(t) &= \left[rS(t)\left(1-\frac{S(t)}{K}\right) - \beta M(t)S(t)f(I) + \delta R(t)\right]dt + \sigma_1 S(t)dW_1(t), \\ dI(t) &= \left[\beta M(t)S(t)f(I) - (\mu+\alpha_1+\lambda+\rho)I(t)\right]dt + \sigma_2 I(t)dW_2(t), \\ dQ(t) &= \left[\lambda I(t) - (\mu+\alpha_2+\varepsilon)Q(t)\right]dt + \sigma_3 Q(t)dW_3(t) \\ dR(t) &= \left[\rho I(t) + \varepsilon Q(t) - (\mu+\delta)R(t)\right]dt + \sigma_4 R(t)dW_4(t), \end{aligned} \tag{3}$$

where $W_i(t)$ is a real-valued standard Brownian motion under the propriety $W_i(0) = 0$ and σ_i represent the intensities of the white noises, with $i = \{1, 2, 3, 4\}$. Next, we define the average

new infections caused by one infected individual. Therefore, applying the next generation method [23], the basic reproduction number can be formulated as follows

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix} I \\ Q \\ R \\ S \end{pmatrix} &= \begin{pmatrix} \beta M(t)S(t)f(I) \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} (\mu+\alpha_1+\lambda+\rho)I(t) \\ (\mu+\alpha_2+\varepsilon)Q(t)-\lambda I(t) \\ (\mu+\delta)R(t)-\rho I(t)-\varepsilon Q(t) \\ \beta M(t)S(t)f(I)-\delta R(t)+rS(t)\left(1-\frac{S(t)}{K}\right) \end{pmatrix} \\ &= \mathcal{F}-\mathcal{V}\end{aligned}$$

The unifected equilibruim point is denoted by $E_0(S_0,0,0,0)$ such that $S_0 = K$ and $M(0) = 1$ we define $D\mathcal{F}$ and $D\mathcal{V}$ as the Jacobian matrices of $\mathcal{F}$ and $\mathcal{V}$ at the point E_0 respectively as

$$D\mathcal{F}(E_0) = \begin{pmatrix} \beta K f'(0) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$D\mathcal{V}(E_0) = \begin{pmatrix} \mu+\alpha_1+\lambda+\rho & 0 & 0 & 0 \\ -\lambda & \mu+\alpha_2+\varepsilon & 0 & 0 \\ -\rho & -\varepsilon & \mu+\delta & 0 \\ \beta K f'(0) & 0 & \delta & 0 \end{pmatrix}$$

Therefore, we select the infection matrix F and the transition matrix V, where

$$F = \begin{pmatrix} \beta K f'(0) & 0 \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} \mu+\alpha_1+\lambda+\rho & 0 \\ -\lambda & \mu+\alpha_2+\varepsilon \end{pmatrix}$$

We obtain the basic reproduction number using the largest eigenvalue of FV^{-1}, with

$$FV^{-1} = \frac{1}{(\mu+\alpha_1+\lambda+\rho)(\mu+\alpha_2+\varepsilon)} \begin{pmatrix} \beta K f'(0) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mu+\alpha_2+\varepsilon & 0 \\ \lambda & \mu+\alpha_1+\lambda+\rho \end{pmatrix}.$$

We obtain that

$$\mathcal{R}_0 = \frac{\beta K f'(0)}{\mu+\alpha_1+\lambda+\rho}, \tag{4}$$

which represents the average number of secondary transmissions of a one infectious individual in an entire vulnerable population.

2.2 Calibration of the model using Moroccan COVID19 data

After the detection of the first case in March 2, 2020 in Casablanca and the succession of new cases of COVID19. The Moroccan authorities began their fight against a risky and a new unknown disease with a declaration,in 13-15 March, of the suspension of all passenger flights and ferry crossings, in 20 March, of a medical state emergency, where authorities require an authorisation from the local state officials for every citizen to leave his home and making exceptions for some workers in essential activities. The media in Morocco played an essential role to the awareness of the population through national TV, social networks and the participation of internet influencers to spread the informations about non pharmaceutical safety measures. However, the easing of the restrictions began in the summer. The country registered a wide spread of infection with a peak of 5415 of daily new cases in 17 November 2020. We list a history of some main events and Moroccan authorities decisions in summer 2020:

- **3,4 July 2020** National attendance for the BAC exam

- **15 July 2020** The beginning of a gradual reopening of mosques with exceptions on holiday and Friday's prays and a safety distance between prayers.
- **22,24 July 2020** National attendance for a retake exam session for the BAC exam.
- **31 July 2020:** Festival of the sacrifice (Eid Al Adha). People gathering in big markets to buy a sacrifice. It represents one of two main holidays celebrated within Islam.
- **6 September 2020:** Moroccan authorities (Ministry for foreign affairs, African cooperation and Moroccan expatriates) open borders with 17 countries and they required a PCR test and a hotel reservation for the access.
- **7 September 2020:** Ministry of national education, professional training, higher education and scientific research declared the beginning of school sessions with an optional choice between distance learning and attendance study. The minister announces that more 80% choose the option of attendance study.
- **7 September 2020:** Closure of schools and the entries of Casablanca as a result of the increase of new cases, closure of markets at 8:00PM and restaurants at 9:00PM, also a curfew at 10:00PM.
- **1,2 and 3 October 2020:** Attendance for the regional BAC exam.
- **5 October 2020:** Attendance study is allowed in Casablanca.

In the literature, many works [26, 27, 28] investigated extensively the COVID19 trending in Morocco. Therefore, we explore in this part a prediction, an estimation of parameters and a calibration the model studied theoretically. Hence, in order to build a numerical approximation, we should define a function $f(I)$ to describe the behavioural changes. Actually, the classical bilinear incidence rate βSI derives from the law of mass action, where it considers the direct contact between infected and susceptible individuals resulting to the spread of the infection. However, it cannot describe a real life transmission for a lot of diseases. In fact, we should take in account the inhibition impact from the conduct of individuals when the infected size increases because of the crowding effect. Thus, the behavioural changes are described by the saturated incidence rate [6, 29] $f(I) = \dfrac{I}{1+rI}$, where r is a positive constant and it is determined in the simulation as $r = 10^{-7}$. In addition, we include a predetermined function to consider the impact of non-pharmaceutical interventions $M(t)$, where we consider the positive constants as $a = 0.96$ and $m = 0.01$ to represent the decisions installed by Moroccan authorities. Although, the positive constants r, a and m were determined to build a scenario to include the decisions and their impact on the behavior of the dynamical system (3).

2.2.1 Data collection:

The dataset of reported daily confirmed cases is collected from the official Coronavirus Portal of Morocco (www.covidmaroc.ma). The total number of data included in this study encloses the cumulative number of reported cases from September 3rd to November 5th 2020. The data is used to adjust the epidemic model to get closer to reality and to estimate the basic reproduction number. The Moroccan ministry of health communicates also many rates such as death rate caused by the infection or the rate of quarantined infected individuals.

3 Results

3.1 Existence of the global and positive solutions

In this section, we generally use the Lyapunov like function [24] to prove that the solution of stochastic epidemic model (3) is positive and global, which it is more general than the monotone or the linear growth conditions to provide the existence of the global and positive solutions.

Theorem 1. *For any initial value* $(S(0), I(0), Q(0), R(0)) \in \mathbb{R}^4_+$*, the stochastic system* (3) *has a unique positive solution* $(S(t), I(t), Q(t), R(t))$ *for all* $t > 0$ *and the solution will remain in* $\mathbb{R}^4_+$ *with probability one. Moreover,* $(S(t), I(t), Q(t), R(t)) \in \mathbb{R}^4_+$ *for all* $t \geq 0$ *a.s.*

Proof. Since the stochastic system (3) has locally Lipschitz coefficients. Then, for any initial value $(S(0), I(0), Q(0), R(0)) \in \mathbb{R}^4_+$, there exists a unique local solution $((S(0), I(0), Q(0), R(0))) \in \mathbb{R}^4_+$ on $t \in [0, \tau_e)$, where τ_e denotes the explosion time. In order to get the global positivity of the solution we need to prove that $\tau_e = \infty$ a.s. Define $k_0 > 0$ to be enough large so that $S(0)$, $I(0)$, $Q(0)$ and $R(0)$ belong to the interval $\left[\frac{1}{k_0}, k_0\right]$. In this matter, for each integer $k \geq k_0$, we consider the following stopping time

$$\tau_k = \inf\left\{t \in [0, \tau_e) : S(t) \notin \left(\frac{1}{k}, k\right) \text{ or } I(t) \notin \left(\frac{1}{k}, k\right) \text{ or } Q(t) \notin \left(\frac{1}{k}, k\right) \text{ or } R(t) \notin \left(\frac{1}{k}, k\right)\right\},$$

where τ_k is increasing as $k \uparrow \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$. Therefore, $\tau_\infty \leq \tau_e$ a.s. Proving that $\tau_\infty = \infty$, means that $\tau_e = \infty$ and $(S(t), I(t), Q(t), R(t)) \in \mathbb{R}^4_+$ a.s. If this statement is false, then there exists a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon$. Thus, there is an integer $k_1 \geq k_0$ such that

$$P\{\tau_k \leq T\} \geq \varepsilon, \quad \forall k \geq k_1. \tag{5}$$

Consider the C^2-function $V_1 : \mathbb{R}^4_+ \to \mathbb{R}_+$ as follows

$$V_1(S, I, Q, R) = (S - 1 - \log S) + (I - 1 - \log I) + (Q - 1 - \log Q) + (R - 1 - \log R).$$

Applying Itô's formula, we obtain

$$\begin{aligned}
\mathcal{L}V_1 &= \left(1 - \frac{1}{S(t)}\right)\left(rS(t)\left(1 - \frac{S(t)}{K}\right) - \beta M(t)S(t)f(I) + \delta R(t)\right) + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} \\
&\quad + \left(1 - \frac{1}{I(t)}\right)(\beta M(t)S(t)f(I) - (\mu + \alpha_1 + \lambda + \rho)I(t)) + \left(1 - \frac{1}{Q(t)}\right)(\lambda I(t) - (\mu + \alpha_2 + \varepsilon)Q(t)) \\
&\quad + \left(1 - \frac{1}{R}\right)(\rho I(t) + \varepsilon Q(t) + (\mu + \delta)R(t)) \\
&\leq rS(t)\frac{(K+1)}{K} - \frac{rS(t)^2}{K} + \beta f(I) - \frac{\delta R(t)}{S(t)} - \frac{\beta M(t)S(t)f(I)}{I(t)} - (\mu + \alpha_1)I(t) - (\mu + \alpha_2)Q(t) \\
&\quad - \lambda\frac{I(t)}{Q(t)} - \varepsilon\frac{Q(t)}{R(t)} - \mu R(t) - r + \mu + \alpha_1 + \lambda + \rho + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} + \mu + \alpha_2 + \varepsilon + \mu + \delta \\
&\leq \sup_{S(t)\in\mathbb{R}_+}\left\{-\frac{rS(t)^2}{K} + rS(t)\frac{K+1}{K}\right\} + 3\mu + \alpha_1 + \alpha_2 + \lambda + \rho + \varepsilon + \delta \\
&\leq \mathcal{K},
\end{aligned} \tag{6}$$

with $\frac{f(I)}{I} \leq f'(0)$ and $\mathcal{K}$ denotes a constant. Hence, the proprieties of a quadratic function is resulting to the boundedness of (6), where integrating its both sides from 0 to $\tau \wedge T$ leads to

$$\begin{aligned}\int_0^{\tau\wedge T} dV_1(S(s),I(s),Q(s),R(s)) \;\leq\; & \int_0^{\tau\wedge T} \mathcal{K}ds + \int_0^{\tau\wedge T} \sigma_1(S(s)-1)dW_1(s) \\ & + \int_0^{\tau\wedge T} \sigma_2(I(s)-1)dW_2(s) + \int_0^{\tau\wedge T} \sigma_3(Q(s)-1)dW_3(s) \\ & + \int_0^{\tau\wedge T} \sigma_4(R(s)-1)dW_4(s). \end{aligned} \tag{7}$$

Taking expectation of both sides of (7), we obtain

$$\mathbb{E}V_1(S(\tau\wedge T),I(\tau\wedge T),Q(\tau\wedge T),R(\tau\wedge T)) \leq V_1(S(0),I(0),Q(0),R(0)) + \mathcal{K}T.$$

This yields to

$$V_1(S(\tau\wedge T),I(\tau\wedge T),Q(\tau\wedge T),R(\tau\wedge T)) + \mathcal{K}T \leq \varepsilon\theta_k, \tag{8}$$

where $\theta_k = (k-1-\log k) \wedge \left(\frac{1}{k} - 1 + \log k\right)$. Letting $k \to \infty$ yields to the contradiction $\infty > V_1(S(0),I(0),Q(0),R(0)) + \mathcal{K}T = \infty$. This finishes the proof. □

Next, we should recall the lemma [24] to show that the solution of the stochastic epidemic model (3) is finite.

Lemma 2. *Let $A(t)$ and $U(t)$ be two continuous adapted increasing process on $t \geq 0$ with $A(0) = U(0) = 0$ a.s. Let $M(t)$ be a real-valued continuous local martingale with $M(0) = 0$ a.s. Let X_0 be a nonnegative $\mathcal{F}$-measurable random variable such that $\mathbb{E}X_0 < \infty$. Define $X(t) = X_0 + A(t) - U(t) + M(t)$ for all $t \geq 0$. If $X(t)$ is nonnegative, then $\lim_{t\to\infty} A(t) < \infty$ implies $\lim_{t\to\infty} U(t) < \infty$, $\lim_{t\to\infty} X(t) < \infty$ and $-\infty < \lim_{t\to\infty} M(t) < \infty$ hold with probability one.*

Theorem 3. *Assume that $(S(t),I(t),Q(t),R(t))$ be a solution of the stochastic system (3) along with initial value $(S(0),I(0),Q(0),R(0)) \in \mathbb{R}^4_+$, then*

$$\lim_{t\to\infty} S(t) + I(t) + Q(t) + R(t) < \infty \quad a.s. \tag{9}$$

Proof. From (1), we get

$$V_1 \leq \xi + M(t), \tag{10}$$

where

$$\xi = V_1(0) + \int_0^t \mathcal{K}ds$$

is a positive $\mathcal{F}_0$-measurable random variable and

$$M(t) = \int_0^t \sigma_1(S-1)dW_1(s) + \sigma_2(I-1)dW_2(s) + \sigma_3(Q-1)dW_3(s) + \sigma_4(R-1)dW_4(s)$$

is a real valued local martingale with $M(0) = 0$ a.s. Therefore, from theorem (1) and lemma (2), we have

$$\limsup_{t\to\infty} V_1(t) < \infty, \quad a.s$$

Notice that $X - 1 - \log X \to \infty$ if and only if $X \uparrow \infty$ or $X \downarrow 0$. By consequent, we get

$$\limsup_{t\to\infty} S(t) < \infty, \quad \limsup_{t\to\infty} I(t) < \infty, \quad \limsup_{t\to\infty} Q(t) < \infty, \quad \limsup_{t\to\infty} R(t) < \infty$$

In consequent, we obtain (9). This finishes the proof. □

According to this result, unlike the deterministic system, adding a noise to the epidemic model (1) will lead the total number of the population to exceed the carrying capacity K.

3.2 Extinction of the disease

In this Section, we explore the extinction of the disease for the stochastic model (3) under some sufficient assumptions. Before, we need to determine the following quantity

$$\mathcal{R}_s = \mathcal{R}_0 - \frac{\sigma_2}{2(\mu+\alpha_1+\lambda+\rho)}, \tag{11}$$

which includes the intensity of the white noise for the infected compartment.

Theorem 4. *Let $(S(t), I(t), Q(t), R(t))$ be the solution of system (3) with any initial value $(S(0), I(0), Q(0), R(0)) \in \mathbb{R}^4_+$. If $R_s < 1$ such that*

$$m := \left(\beta f'(0)\int_0^\infty \pi(x)x dx + (\mu+\alpha_1+\lambda+\rho)(\mathcal{R}_s - 1)\right) < 0,$$

where $x \in (0,\infty)$ and $\pi = Qx^{-2+\frac{r}{\sigma_1^2}} e^{-\frac{2}{\sigma^2}\left((\frac{rx}{K}+\frac{\delta C}{x})\right)}$, where Q is constant such that $\int_0^\infty \pi(x)dx = 1$ then the disease of system (3) will go to extinction almost surely.

$$\lim_{t\to\infty} I(t) = 0, \quad a.s.$$

and

$$\lim_{t\to\infty}\langle Q(t)\rangle = \lim_{t\to\infty}\langle R(t)\rangle = 0 \ a.s. \tag{12}$$

Moreover, the process $S(t)$ converges in distribution to the invariant measure m in $\mathbb{R}^+$ which has the density $\pi(x)$.

Proof. For any initial value $(S(0), I(0), Q(0), R(0)) \in \mathbb{R}^4_+$, we have a positive solution for the system (3). Therefore, we have

$$dS(t) \le \left[rS(t)\left(1-\frac{S(t)}{K}\right) + \delta K\right]dt + \sigma_1 S(t) dW_1(t). \tag{13}$$

Next, we consider the following stochastic logistic equation

$$h(x) = rx\left(1-\frac{x}{K}\right) + \delta C, \quad \sigma(x) = x\sigma_1(x), \quad x \in (0,\infty), \tag{14}$$

where, according the theorem (3), we determine a constant C greater than K. Hence, we get that

$$\begin{aligned}\int \frac{h(s)}{\sigma^2(s)}ds &= \frac{1}{\sigma^2}\int \frac{r\left(1-\frac{x}{K}\right)}{x} + \frac{\delta C}{x^2}ds \\ &= \frac{1}{\sigma^2}\left[r\log x - \frac{rx}{K} - \frac{\delta C}{x}\right] + Q \end{aligned} \tag{15}$$

In consequent, we get

$$e^{\int \frac{h(s)}{\sigma^2(s)}ds} = e^Q x^{\frac{r}{\sigma_1^2}} e^{-\frac{1}{\sigma_1^2}(\frac{rx}{K}+\frac{\delta C}{x})} \tag{16}$$

Hence, we obtain

$$\int_0^\infty \frac{1}{\sigma^2(x)} e^{\int_1^x \frac{2h(\tau)}{\sigma^2(\tau)}d\tau}dx = \int_0^\infty x^{-2}x^{\frac{r}{\sigma_1^2}}e^{-\frac{2}{\sigma^2}\left((\frac{rx}{K}+\frac{\delta C}{x})\right)}dx < \infty. \tag{17}$$

Considering the ergodic propriety and the invariant density, we have

$$\pi(x) = Qx^{-2+\frac{r}{\sigma_1^2}} e^{-\frac{2}{\sigma^2}\left(\left(\frac{rx}{K}+\frac{\delta C}{x}\right)\right)}, \text{ where Q is constant such that } \int_0^\infty \pi(x)dx = 1. \tag{18}$$

Therefore

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds = \int_0^\infty \pi(x)xdx \quad \text{a.s.} \tag{19}$$

By the comparison theorem, we get $S(t) \leq x(t)$ for any $t \leq 0$ a.s. In the follow, we apply Itô's formula on $\log I(t)$. Therefore, we get

$$\begin{aligned} d\log(I(t)) &= \left[\beta M(t)S(t)\frac{f(I)}{I(t)} - (\mu+\alpha_1+\lambda+\rho) - \frac{\sigma_2^2}{2}\right]dt + \sigma_2 dW_2(t) \\ &\leq \left[\beta X(t)f'(0) - (\mu+\alpha_1+\lambda+\rho) - \frac{\sigma_2^2}{2}\right]dt + \sigma_2 dW_2(t) \\ &\leq \left[\beta f'(0)(X(t)-K) + \beta f'(0)K - (\mu+\alpha_1+\lambda+\rho) - \frac{\sigma_2^2}{2}\right]dt + \sigma_2 dW_2(t) \end{aligned} \tag{20}$$

Integrating both sides of (20) from 0 to t, we get

$$\frac{\log I(t)}{t} \leq \left(\beta f'(0)\int_0^t |x(t)-K|ds + (\mu+\alpha_1+\lambda+\rho)(\mathcal{R}_s - 1)\right) + \frac{\log I(0)}{t} + \frac{\sigma_2 W(t)}{t} \tag{21}$$

It follows from the ergodic proprieties of $x(t)$ and $\int_0^\infty x\pi(x)dx < \infty$ that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t |x(s)-K|ds = \int_0^\infty |x(s)-K|\pi(x)dx < \infty. \tag{22}$$

By the law of large number for martingales, and for $\mathcal{R}_s < 1$, we get

$$\limsup_{t\to\infty}\frac{\log I(t)}{t} \leq \left(\beta f'(0)\int_0^\infty \pi(x)xdx + (\mu+\alpha_1+\lambda+\rho)(\mathcal{R}_s-1)\right) < 0 \text{ a.s,} \tag{23}$$

which leads to

$$\lim_{0\to\infty} I(t) = 0 \text{ a.s.} \tag{24}$$

From the quarantine compartment of system (3) we deduce that

$$\frac{Q(t)-Q(0)}{t} = \lambda\langle I(t)\rangle - (\mu+\alpha_2+\varepsilon)\langle Q(t)\rangle + \frac{\sigma_3}{t}\int_0^t Q(t)dW_3(s). \tag{25}$$

Therefore for theorem (3), the strong law of large numbers for martingales and the equations (24), (25) and by the same step for the fourth equation in the stochastic system (3), we get

$$\lim_{t\to\infty}\langle Q(t)\rangle = \lim_{t\to\infty}\langle R(t)\rangle = 0 \text{ a.s.} \tag{26}$$

This finishes the proof. □

3.3 Positive recurrence and ergodic properties of system (3)

In this section, based on the theory of Has'minskii [25], we verify that there is an ergodic stationary distribution, which reveals that the infection will persist. Here we present some theory about the stationary distribution (see Has'minskii [25]). Let $X(t)$ be a homogeneous

Markov process in E_n (E_n represents n-dimensional Euclidean space), and is described by the following stochastic differential equation

$$dX(t) = b(X)dt + \sum_{i=1}^{k} g_i(X)dW_i(t). \tag{27}$$

The diffusion matrix is defined as follows

$$\mathbb{A}(x) = (\Lambda_{ij}(x)), \qquad \Lambda_{ij}(x) = \sum_{i=1}^{k} g_k^i(x)g_k^j(x). \tag{28}$$

Lemma 5 ([25]). *The Markov process $X(t)$ has a unique ergodic stationary distribution $\mu(.)$ if there exists a bounded domain $D \subset E_n$ with regular boundary Γ and*

- (H_1)*: there is a positive number M such that* $\sum_{i=1}^{k}(\Lambda_{ij}(x))\varepsilon_i\varepsilon_j \geq M|\epsilon|^2$, $x \in D, \varepsilon \in \mathbb{R}^n$.
- (H_2)*: there exists a nonnegative $C-$function V such that $\mathcal{L}V$ is negative for any $E_n \setminus D$.*

$$\mathbb{P}\left\{\lim_{T\to T}\frac{1}{T}\int_0^T f(X(t))dt = \int_{E_n} f(x)\mu(dx)\right\} = 1$$

Then for all $x \in E_n$, where $f(.)$ is a function integrable with respect to the measure μ.

Theorem 6. *Assume that $\tilde{\mathcal{R}}_s > 1$. Then, the stochastic system* (3) *has a unique stationary distribution $\mu(.)$ and it has the ergodic property.*

Proof. We determine the diffusion matrix of the stochastic epidemic model with logistic growth (3) as

$$\mathcal{A}(S,I,Q,R) = \begin{bmatrix} \sigma_1^2S^2 & 0 & 0 & 0 \\ 0 & \sigma_2^2I^2 & 0 & 0 \\ 0 & 0 & \sigma_3^2Q^2 & 0 \\ 0 & 0 & 0 & \sigma_4^2R^2 \end{bmatrix}. \tag{29}$$

Next, we consider Γ to be any bounded domain in $\mathbb{R}^4_+$. Therefore, there exists a positive constant

$$L_0 = \min\{\sigma_1^2S^2, \sigma_2^2I^2, \sigma_3^2Q^2, \sigma_4^2R^2, (S,I,Q,R) \in \bar{\Gamma}\},$$

where

$$\begin{aligned} \Sigma_{i,j=1}^3 a_{ij}(S,I,Q,R)\xi_i\xi_j &= \sigma_1^2S^2\xi_1^2 + \sigma_2^2I^2\xi_2^2 + \sigma_3^2Q^2\xi_3^2 + \sigma_4^2R^2\xi_4^2 \\ &\geq L_0|\xi|^2, (S,I,Q,R) \in \bar{\Gamma}_\sigma, \xi = (\xi_1,\xi_2,\xi_3,\xi_4) \in \mathbb{R}^4. \end{aligned}$$

This yields to the first condition to be verified, with the smallest eigenvalue of the diffusion matrix $\mathcal{A}(S,I,Q,R)$ is bounded away from zero.
In the follow, let $\tilde{V}(X_t,t)$ be a C^2- function with $X(t) = (S(t),I(t),Q(t),R(t))$. Also, a closed set $U_\varepsilon \in \mathbb{R}^4_+$ such as $\sup_{X\in\mathbb{R}^4_+\setminus U_\varepsilon} \mathcal{L}\tilde{V} < -\tilde{M} < 0$, where $\tilde{M}$ denotes a positive constant and

$$\mu - \frac{m(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)}{2} > 0, \tag{30}$$

where m is a positive constant.
We construct a Lyapunov functional such that

$$\hat{V}(X_t,t) = \vartheta\tilde{V}_1 + \tilde{V}_2 + \tilde{V}_3, \tag{31}$$

with

$$\begin{aligned}\tilde{V}_1 &= -\left(\frac{\beta K f'(0)}{r}\log(S(t)) + \log(I(t))\right),\\ \tilde{V}_2 &= -\log S - \log Q - \log R,\\ \tilde{V}_3 &= \frac{1}{m+1}\left(S+I+Q+R\right)^{m+1},\end{aligned}$$

where $\vartheta > 0$ is considered as a large enough constant verifying

$$-\vartheta\tilde{\lambda} + D \leq -2. \tag{32}$$

Afterwards, we define the terms $\tilde{\lambda}$ and D.
Furthermore, $\tilde{V}(X_t, t)$ is a continuous function with minimum point (S_0, I_0, Q_0, R_0) in the interior of $\mathbb{R}^4_+$. Hence, we consider $\tilde{V} : \mathbb{R}^4_+ \to \mathbb{R}_+$ to be a nonnegative function such that

$$\tilde{V} = \hat{V}(S, I, Q, R) - \hat{V}(S_0, I_0, Q_0, R_0).$$

Applying Itô's formula on $\tilde{V}_1$, we get

$$\begin{aligned}\mathcal{L}\tilde{V}_1 &= \frac{\beta K f'(0)}{r}\left\{-r\left(1-\frac{S(t)}{K}\right) + \beta M(t) f(I) - \delta\frac{R(t)}{S(t)} + \frac{\sigma_1^2}{2}\right\} - \beta M(t)\frac{f(I)}{I}S(t)\\ &\quad +\mu + \alpha_1 + \lambda + \rho + \frac{\sigma_2^2}{2}\\ &= -\beta K f'(0) + \beta f'(0)S(t) + \frac{\beta^2 K f'(0)}{r} f(I) - \frac{\delta\beta K f'(0) R(t)}{rS(t)} + \frac{\beta K f'(0)\sigma_1^2}{2r}\\ &\quad -\beta M(t)\frac{f(I)}{I}S(t) + \mu + \alpha_1 + \lambda + \rho + \frac{\sigma_2^2}{2}\\ &\leq -\left(\beta K f'(0) - \left(\mu + \alpha_1 + \lambda + \rho + \frac{\sigma_2^2}{2} + \frac{\beta K f'(0)\sigma_1^2}{2r}\right)\right) + \beta S(t)\left(f'(0) - M(t)\frac{f(I)}{I}\right)\\ &\quad +\frac{\beta^2 K f'(0)^2}{r} I.\end{aligned} \tag{33}$$

From theorem (3), and the monotonicity and Lipschitz assumptions of $\frac{f(.)}{.}$. for $x < y \in \mathbb{R}_+$ and a constant $\mathcal{C}$, we have

$$\frac{f(x)}{x} - \frac{f(y)}{y} \leq \mathcal{C}(y - x). \tag{34}$$

Extending x to zero, we get

$$f'(0) - \frac{f(I)}{I} \leq \mathcal{C}I \quad \text{for any } I \in \mathbb{R}_+.$$

Hence, we obtain

$$\begin{aligned}\mathcal{L}\tilde{V}_1 &\leq -(\mu + \alpha_1 + \lambda + \rho)(\tilde{\mathcal{R}}_s - 1) + \mathcal{C}\beta S(t)I(t) + \beta(1 - M(t))\frac{f(I)}{I}S(t) + \frac{\beta^2 K f'(0)^2}{r} I\\ &\leq -\tilde{\gamma} + \left[\mathcal{C}^2 + \frac{\beta f'(0)}{r}\right]\beta K I(t) + \beta f'(0)S(t),\end{aligned} \tag{35}$$

where $\tilde{\mathcal{R}}_s = \mathcal{R}_s - \dfrac{\beta K f'(0)}{2r(\mu + \alpha_1 + \lambda + \rho)}\sigma_1^2$ and $\tilde{\gamma} > 0$.
Applying Itô's formula on $\tilde{V}_2$, we get

$$\begin{aligned}\mathcal{L}\tilde{V}_2 &= -r\left(1 - \frac{S(t)}{K}\right) + \beta M(t) f(I) - \delta\frac{R(t)}{S(t)} + \frac{\sigma_1^2}{2} - \lambda\frac{I(t)}{Q(t)}\\ &\quad +\mu + \alpha_2 + \varepsilon - \rho\frac{I(t)}{R(t)} - \varepsilon\frac{Q(t)}{R(t)} + \mu + \delta.\end{aligned} \tag{36}$$

After, we use Itô's formula on $\tilde{V}_3$, with m is a constant satisfying the following condition (30), we obtain

$$\begin{aligned}
\mathcal{L}\tilde{V}_3 &= (S(t)+I(t)+Q(t)+R(t))^m\left(rS(t)\left(1-\frac{S(t)}{K}\right)-(\mu+\alpha_1)I(t)-(\mu+\alpha_2)Q(t)-\mu R(t)\right)\\
&\quad+\frac{m}{2}(S(t)+I(t)+Q(t)+R(t))^{m-1}(\sigma_1^2S+\sigma_2^2I+\sigma_3^2Q+\sigma_4^2R)\\
&\leq (S(t)+I(t)+Q(t)+R(t))^m\left(rS(t)\left(1-\frac{S(t)}{K}\right)-(\mu+\alpha_1)I(t)-(\mu+\alpha_2)Q(t)-\mu R(t)\right)\\
&\quad+\frac{m}{2}(S(t)+I(t)+Q(t)+R(t))^{m+1}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)\\
&\leq rS(S(t)+I(t)+Q(t)+R(t))^m-\frac{r}{K}S^{m+2}-\mu(I^{m+1}+Q^{m+1}+R^{m+1})\\
&\quad+\frac{m}{2}(S(t)+I(t)+Q(t)+R(t))^{m+1}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)
\end{aligned}$$

Using the inequality $|\sum_{i=1}^{k}a_i|^n\leq k^{n-1}\sum_{i=1}^{k}|a_i|^n$, $\forall n\geq 1$, we get

$$\begin{aligned}
\mathcal{L}\tilde{V}_3 &\leq -\frac{r}{2K}S^{m+2}-\frac{\mu+\alpha_1}{2}I^{m+1}-\frac{\mu+\alpha_2}{2}Q(t)^{m+1}-\frac{\mu}{2}R(t)^{m+1}-\frac{r}{2K}S^{m+2}\\
&\quad-\frac{\mu+\alpha_1}{4}I^{m+1}-\frac{\mu+\alpha_2}{4}Q(t)^{m+1}-\frac{\mu}{4}R(t)^{m+1}+rS(t)(S(t)+I(t)+Q(t)+R(t))^m\\
&\quad+\frac{4^mm}{2}S(t)^{m+1}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)-\left(\frac{\mu+\alpha_1}{4}-\frac{4^mm}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)\right)I(t)^{m+1}\\
&\quad-\left(\frac{\mu+\alpha_2}{4}-\frac{4^mm}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)\right)Q(t)^{m+1}-\left(\frac{\mu}{4}-\frac{4^mm}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)\right)R(t)^{m+1}\\
&\leq -\frac{r}{2K}S^{m+2}-\frac{\mu+\alpha_1}{2}I^{m+1}-\frac{\mu+\alpha_2}{2}Q(t)^{m+1}-\frac{\mu}{2}R(t)^{m+1}+A, \qquad (37)
\end{aligned}$$

where

$$\begin{aligned}
A=\sup_{(S,I,Q,R)\in\mathbb{R}^4_+}\quad&\left\{-\frac{r}{2K}S^{m+2}-\frac{\mu+\alpha_1}{4}I^{m+1}-\frac{\mu+\alpha_2}{4}Q^{m+1}-\frac{\mu}{4}R^{m+1}+rS(S+I+Q+R)^m\right.\\
&\left.+\frac{4^mm}{2}S(t)^{m+1}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)\right\}.
\end{aligned}$$

From (35), (36) and (37), we obtain

$$\begin{aligned}
\mathcal{L}\tilde{V} &\leq -\vartheta\tilde{\gamma}+\left[\mathcal{C}^2+\frac{\beta f'(0)}{r}\right]\beta KI(t)+\beta f'(0)S(t)+r\frac{S(t)}{K}+\beta M(t)f'(0)I+\frac{\sigma_1^2}{2}\\
&\quad+2\mu+\alpha_2+\varepsilon+\delta-\frac{r}{2K}S^{m+2}-\frac{\mu+\alpha_1}{2}I^{m+1}-\frac{\mu+\alpha_2}{2}Q(t)^{m+1}\\
&\quad-\frac{\mu}{2}R(t)^{m+1}+A. \qquad (38)
\end{aligned}$$

In the follow, we consider a compact subset U_ϵ such that (H_2) is fulfilled and let the bounded closed set U_ϵ as follows

$$U_\epsilon=\left\{(S,I,Q,R)\in\mathbb{R}^4_+,\quad \epsilon^3\leq S\leq\frac{1}{\epsilon^3},\quad \epsilon\leq I\leq\frac{1}{\epsilon},\quad \epsilon^3\leq Q\leq\frac{1}{\epsilon^3},\quad \epsilon^3\leq R\leq\frac{1}{\epsilon^3}\right\},$$

with $0<\epsilon<1$ is a sufficiently small such that

$$-\frac{\delta}{\epsilon}+B<-1, \tag{39}$$
$$-\nu\tilde{\gamma}+\beta\left[\mathcal{C}^2K+\frac{Kf'(0)}{r}+f'(0)\right]\epsilon+E<-1, \tag{40}$$
$$-\frac{\lambda}{\epsilon}+F<-1, \tag{41}$$
$$-\frac{\rho}{\epsilon}+G<-1, \tag{42}$$
$$-\frac{r}{4K\epsilon^{3m+6}}+H<-1, \tag{43}$$
$$-\frac{\mu+\alpha_1}{4\epsilon^{m+1}}+I<-1, \tag{44}$$
$$-\frac{\mu+\alpha_2}{4\epsilon^{m+1}}+J<-1, \tag{45}$$
$$-\frac{\mu}{4\epsilon^{m+1}}+N<-1, \tag{46}$$

with B, E, F, G, H, I, J and n are positive constants, where the expressions are derived from the presented cases for a sufficiently small ϵ, where $0<\epsilon<1$. Hence, we divide $\mathbb{R}^4_+\backslash U_\varepsilon$ into eight domains, such that

$$U_1=\{(S,I,Q,R)\in\mathbb{R}^4_+, 0<S<\epsilon^3,\quad \epsilon^2\leq R<\epsilon\},\quad U_2=\{(S,I,Q,R)\in\mathbb{R}^4_+, 0<I<\epsilon\},$$
$$U_3=\{(S,I,Q,R)\in\mathbb{R}^4_+, \epsilon^2\leq I<\epsilon,\quad 0<Q<\epsilon^3\},$$
$$U_4=\{(S,I,Q,R)\in\mathbb{R}^4_+, \epsilon^2\leq I<\epsilon,\quad 0<R<\epsilon^3\},$$
$$U_5=\left\{(S,I,Q,R)\in\mathbb{R}^4_+, S>\frac{1}{\epsilon}\right\},\quad U_6=\left\{(S,I,Q,R)\in\mathbb{R}^4_+, I>\frac{1}{\epsilon}\right\},$$
$$U_7=\left\{(S,I,Q,R)\in\mathbb{R}^4_+, Q>\frac{1}{\epsilon^3}\right\},\quad U_8=\left\{(S,I,Q,R)\in\mathbb{R}^4_+, R>\frac{1}{\epsilon^3}\right\}.$$

Knowing that $U_\epsilon^c=U_1\cup U_2\cup U_3\cup U_4\cup U_5\cup U_6\cup U_7\cup U_8$. Next, we should show that $\mathcal{L}\tilde{V}\leq -1$ on $\mathbb{R}^3_+\backslash U_\epsilon$, it means fulfilling it on the above eight domains.

Case 1: $0<S<\epsilon^3$, $\epsilon^2\leq R<\epsilon$

$$\begin{aligned}\mathcal{L}\tilde{V} &\leq -\frac{\delta}{\epsilon}+\left[\beta f'(0)+\frac{r}{K}\right]S(t)+\left[\mathcal{C}^2+\frac{f'(0)}{r}\right]\beta K+M(t)f'(0)I(t)+\frac{\sigma_1^2}{2}+2\mu+\alpha_2\\ &\quad \varepsilon+\delta-\frac{r}{2K}S(t)^{m+2}-\frac{\mu+\alpha_1}{2}I(t)^{m+1}-\frac{\mu+\alpha_2}{2}Q(t)^{m+1}-\frac{\mu}{2}R(t)^{m+1}+A\\ &\leq -\frac{\delta}{\epsilon}+B.\end{aligned} \tag{47}$$

$$\begin{aligned}B &= \sup_{(S,I,Q,R)\in\mathbb{R}^4_+}\left\{\left[\beta f'(0)+\frac{r}{K}\right]S(t)+\left[\mathcal{C}^2+\frac{f'(0)}{r}\right]\beta K+M(t)f'(0)I(t)+\frac{\sigma_1^2}{2}+2\mu+\alpha_2\right.\\ &\quad \left.\varepsilon+\delta-\frac{r}{2K}S(t)^{m+2}-\frac{\mu+\alpha_1}{2}I(t)^{m+1}-\frac{\mu+\alpha_2}{2}Q(t)^{m+1}-\frac{\mu}{2}R(t)^{m+1}+A\right\}.\end{aligned} \tag{48}$$

According to (39), we get that $\mathcal{L}\tilde{V}\leq -1$ for any $(S,I,Q,R)\in U_1$.

Case 2: $0<I<\epsilon$.

$$\mathcal{L}\tilde{V} \leq -\nu\tilde{\gamma}+\beta\left[\mathcal{C}^2K+\frac{Kf'(0)}{r}+f'(0)\right]\varepsilon+E. \tag{49}$$

$$E = \sup_{(S,I,Q,R)\in\mathbb{R}^4_+} \left\{ \left[\beta f'(0) + \frac{r}{K}\right] S(t) + \frac{\sigma_1^2}{2} + 2\mu + \alpha_2 + \varepsilon - \frac{r}{2K} S^{m+2} - \frac{(\mu+\alpha_1)}{2} I^{m+1} - \frac{\mu+\alpha_2}{2} Q^{m+1} - \frac{\mu}{2} R^{m+1} + A \right\}. \quad (50)$$

By Virtue of (40), we obtain that $\mathcal{L}\tilde{V} \leq -1$ for any $(S, I, Q, R) \in U_2$.

Case 3: $0 < Q < \epsilon^3$ **and** $\epsilon^2 \leq I < \epsilon$**.**

$$\mathcal{L}\tilde{V} \leq -\frac{\lambda}{\epsilon} + F. \quad (51)$$

$$F = \sup_{(S,I,Q,R)\in\mathbb{R}^4_+} \left\{ \left[\beta f'(0) + \frac{r}{K}\right] S(t) + \frac{\sigma_1^2}{2} + 2\mu + \alpha_2 + \varepsilon - \rho\frac{I(t)}{R(t)} - \varepsilon\frac{Q(t)}{R(t)} - \delta\frac{R(t)}{S(t)} - \frac{r}{2K} S^{m+2} - \frac{(\mu+\alpha_1)}{2} I^{m+1} - \frac{\mu+\alpha_2}{2} Q^{m+1} - \frac{\mu}{2} R^{m+1} + A \right\}. \quad (52)$$

It follows from (41) that $\mathcal{L}\tilde{V} \leq -1$ for any $(S, I, Q, R) \in U_3$.

Case 4: $0 < R < \epsilon^3$ **and** $\epsilon^2 \leq I < \epsilon$**.**

$$\mathcal{L}\tilde{V} \leq -\frac{\rho}{\epsilon} + G. \quad (53)$$

$$G = \sup_{(S,I,Q,R)\in\mathbb{R}^4_+} \left\{ \left[\beta f'(0) + \frac{r}{K}\right] S(t) + \frac{\sigma_1^2}{2} + 2\mu + \alpha_2 + \varepsilon - \varepsilon\frac{Q(t)}{R(t)} - \delta\frac{R(t)}{S(t)} - \lambda\frac{I(t)}{Q(t)} - \frac{r}{2K} S^{m+2} - \frac{(\mu+\alpha_1)}{2} I^{m+1} - \frac{\mu+\alpha_2}{2} Q^{m+1} - \frac{\mu}{2} R^{m+1} + A \right\}. \quad (54)$$

By virtue of (41), we conclude that $\mathcal{L}\tilde{V} \leq -1$ for any $(S, I, Q, R) \in U_4$.

Case 5: $S > \dfrac{1}{\epsilon^3}$**.**

$$\mathcal{L}\tilde{V} \leq -\frac{r}{4K\varepsilon^{3m+6}} + H. \quad (55)$$

$$H = \sup_{(S,I,Q,R)\in\mathbb{R}^4_+} \left\{ \left[\beta f'(0) + \frac{r}{K}\right] S(t) + \frac{\sigma_1^2}{2} + 2\mu + \alpha_2 + \varepsilon - \rho\frac{I(t)}{R(t)} - \varepsilon\frac{Q(t)}{R(t)} - \delta\frac{R(t)}{S(t)} - \lambda\frac{I(t)}{Q(t)} - \frac{r}{4K} S^{m+2} - \frac{(\mu+\alpha_1)}{2} I^{m+1} - \frac{\mu+\alpha_2}{2} Q^{m+1} - \frac{\mu}{2} R^{m+1} + A \right\}. \quad (56)$$

According to (43), we get that $\mathcal{L}\tilde{V} \leq -1$ for any $(S, I, Q, R) \in U_5$.

Case 6: $I > \dfrac{1}{\epsilon}$**.**

$$\mathcal{L}\tilde{V} \leq -\frac{\mu+\alpha_1}{4\varepsilon^{m+1}} + I. \quad (57)$$

$$I = \sup_{(S,I,Q,R)\in\mathbb{R}^4_+} \left\{ \left[\beta f'(0) + \frac{r}{K}\right] S(t) + \frac{\sigma_1^2}{2} + 2\mu + \alpha_2 + \varepsilon - \rho\frac{I(t)}{R(t)} - \varepsilon\frac{Q(t)}{R(t)} - \delta\frac{R(t)}{S(t)} - \lambda\frac{I(t)}{Q(t)} - \frac{r}{2K} S^{m+2} - \frac{(\mu+\alpha_1)}{4} I^{m+1} - \frac{\mu+\alpha_2}{2} Q^{m+1} - \frac{\mu}{2} R^{m+1} + A \right\}. \quad (58)$$

It follows from (44) that $\mathcal{L}\tilde{V} \leq -1$ for any $(S, I, Q, R) \in U_6$.

Case 7: $Q > \dfrac{1}{\epsilon^3}$.

$$\mathcal{L}\tilde{V} \leq -\frac{\mu+\alpha_2}{4\varepsilon^{m+1}} + J. \tag{59}$$

$$J = \sup_{(S,I,Q,R)\in\mathbb{R}^4_+} \left\{ \left[\beta f'(0) + \frac{r}{K}\right] S(t) + \frac{\sigma_1^2}{2} + 2\mu + \alpha_2 + \varepsilon - \rho\frac{I(t)}{R(t)} - \varepsilon\frac{Q(t)}{R(t)} - \delta\frac{R(t)}{S(t)} - \lambda\frac{I(t)}{Q(t)} - \frac{r}{2K}S^{m+2} - \frac{(\mu+\alpha_1)}{2}I^{m+1} - \frac{\mu+\alpha_2}{4}Q^{m+1} - \frac{\mu}{2}R^{m+1} + A \right\}. \tag{60}$$

In view of (45), we get that $\mathcal{L}\tilde{V} \leq -1$ for any $(S, I, Q, R) \in U_7$.

Case 8: $R > \dfrac{1}{\epsilon^3}$.

$$\mathcal{L}\tilde{V} \leq -\frac{\mu}{4\varepsilon^{m+1}} + N. \tag{61}$$

$$N = \sup_{(S,I,Q,R)\in\mathbb{R}^4_+} \left\{ \left[\beta f'(0) + \frac{r}{K}\right] S(t) + \frac{\sigma_1^2}{2} + 2\mu + \alpha_2 + \varepsilon - \rho\frac{I(t)}{R(t)} - \varepsilon\frac{Q(t)}{R(t)} - \delta\frac{R(t)}{S(t)} - \lambda\frac{I(t)}{Q(t)} - \frac{r}{2K}S^{m+2} - \frac{(\mu+\alpha_1)}{2}I^{m+1} - \frac{\mu+\alpha_2}{2}Q^{m+1} - \frac{\mu}{4}R^{m+1} + A \right\}. \tag{62}$$

Therefore, from (46), we obtain that $\mathcal{L}\tilde{V} \leq -1$. $(S, I, Q, R) \in \mathbb{R}^4_+$.Hence, It follows from (47), (49), (51), (53), (55), (57), (59) and (61), we get for a sufficiently small ϵ that

$$\mathcal{L}\tilde{V} \leq -1 \text{ for all } (S, I, Q, R) \in \mathbb{R}^4_+ \backslash U_\epsilon \tag{63}$$

Therefore, the assumption H_2 in Lemma (5) is fulfilled. By Lemma (5) , we can claim that stochastic epidemic model (3) is ergodic and admits a unique stationary distribution. The proof is completed. □

3.3.1 Model calibration:

In this part of investigation, we consider a set of COVID19 cases data in Morocco denoted as $Y_T \triangleq \{y_0, y_1, ..., y_T\}$, where the observation is up to a finite horizon T and $y_t \triangleq [S_t, I_t, Q_t, R_t]'$ is a column vector in $\mathbb{R}^{4\times 1}$, which it represents at time t the daily observed values for the susceptible "S", infected "I", quarantined "Q" and recovered "R" individuals. Therefore, we estimate the unknown parameters in the epidemic model, described as a column vector $\theta \triangleq [\beta, \rho, \lambda, \varepsilon]'$. For this matter, we describe the predicted epidemic model $\hat{y}_t(\theta)$ such as

$$\hat{y}_t(\theta) \triangleq \begin{bmatrix} S_{t-1} + rS_{t-1}\left(1 - \dfrac{S_{t-1}}{K}\right) + \beta M_{t-1}\dfrac{S_{t-1}I_{t-1}}{1+rI_{t-1}} + \delta R_{t-1} \\ I_{t-1} + \beta M_{t-1}\dfrac{S_{t-1}I_{t-1}}{1+rI_{t-1}} - (\mu+\alpha_1+\lambda+\rho)I_{t-1} \\ Q_{t-1} + \lambda I_{t-1} - (\mu+\alpha_2+\varepsilon)Q_{t-1} \\ R_{t-1} + \rho I_{t-1} + \varepsilon Q_{t-1} - (\mu+\delta)R_{t-1} \end{bmatrix}, \tag{64}$$

where $\hat{y}_0(\theta) = y_0$ as an initial condition at time 0. Using the Euler method to approximate the solution, we calculate the quadratic cost

$$J_T(\theta) \triangleq \frac{1}{2}\sum_{t=0}^{T} ||y_t - \hat{y}_t(\theta)||^2. \tag{65}$$

The minimum of the quadratic cost J in order to get the least square estimator θ_e, where

$$\theta_e \triangleq \underset{\theta \in \mathbb{R}^4}{\operatorname{argmin}} \quad J_T(\theta) \tag{66}$$

Hence, to implement real dataset, we use a 64 days data of Moroccan COVID19 cases to calibrate it with the deterministic model using least squares method and Euler approximation to the model (3). As result we get the following values

Therefore, the basic reproduction number (4) for cases generated by one case is estimated

Parameter	Description	Estimation	Source
r	Intrinsic growth rate	3.33×10^{-5}	world population review
K	Carrying capacity	47800000	world population review
β	Transmission coefficient	5.21×10^{-8}	Estimated
μ	Natural death rate	1.4167×10^{-5}	Data World Bank
α_1	Death rate for infected individuals	0.018	COVIDMAROC
α_2	Death rate for quarantined individuals	0.018	COVIDMAROC
ρ	Recovery rate for infected individuals	0.72	Estimated
ϵ	Recovery rate for quarantined individuals	0.57	Estimated
λ	Quarantined rate for infected individuals	0.96	Estimated
δ	Loss of immunity rate for recovered individuals	0.04	Assumed

Table 1: Table of parameters used in the numerical simulation and estimated by the data fitting.

as 1.46. As result, it means the persistence of the infectious disease in Morocco despite the measures taken as non-pharmaceutical interventions to slow down its spread. Up to date, the daily COVID19 confirmed cases continues to rise and fluctuate on a regular basis.

3.4 Numerical simulations and discussions

Data fitting and parameter estimation is a critical phase in this part of investigation to support our theoretical results. In figure (1), we illustrate the optimization of the best fit to the provided Moroccan dataset since it confirms the observed dynamics of the infection during the pandemic outbreak. We provide a numerical approximation using estimated parameters $\theta_e = [5.21 \times 10^{-8} \quad 0.72 \quad 0.96 \quad 0.57]'$ by the least squares method. Here, we compute using the initial condition $y_0 = [36404398 \quad 5895 \quad 8483 \quad 51223]$. The total population of Morocco is provided by World bank, while the infected, quarantined and recovered size is communicated by Moroccan ministry of health throughout COVIDMAROC. In different circumstances, in order to get closer to the reality and to data collected in autumn, where the environment is largely associated to fluctuations in temperature, winds, humidity and drop of immunity, also some unreported cases. An addition of a noise is desirable to each compartment specially the infected and quarantined individuals. Therefore, to include the environmental noise, we illustrate the stochastic epidemic model (3) and its behaviour, where we determine the values the volatilities as $\sigma_1 = 7.86 \times 10^{-5}, \sigma_2 = 0.17, \sigma_3 = 0.21$ and $\sigma_4 = 2.2 \times 10^{-2}$. Indeed, as illustrated in figures (2) and (3), the red range denotes the stochastic scenarios that can be generated respectively by σ_2 and σ_3 including the fluctuations of Moroccan daily confirmed cases.

Hence, from the collection of 64 days data, we seek to predict the following 15 days using the solutions of the stochastic epidemic model (3). As result, the range of predicted possible solutions expands due to the increase of infected and quarantined cases since the diffusion part

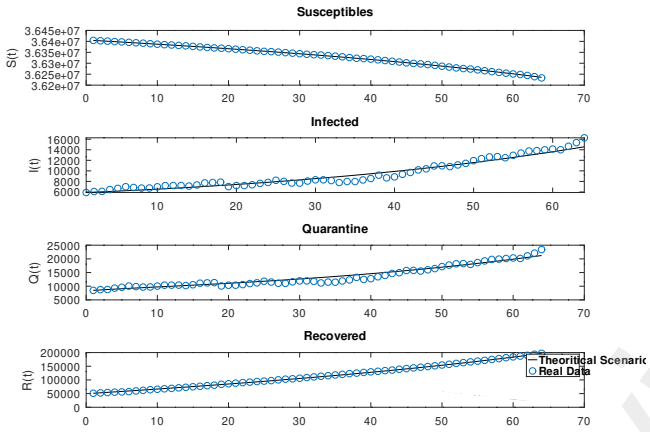

Figure 1: Trajectories of calibrated model with real data of Moroccan COVID19 cases.

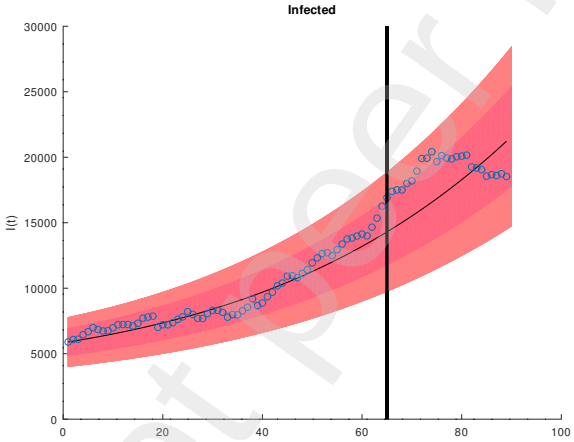

Figure 2: Trajectories of infected cases of COVID19 in Morocco and the simulated model.

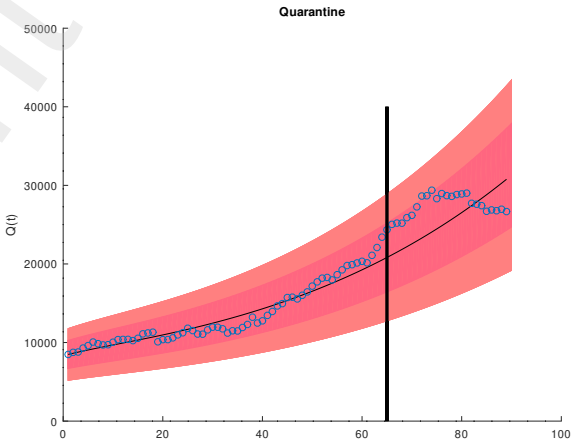

Figure 3: Trajectories of quarantined individuals of COVID19 in Morocco and the simulated model.

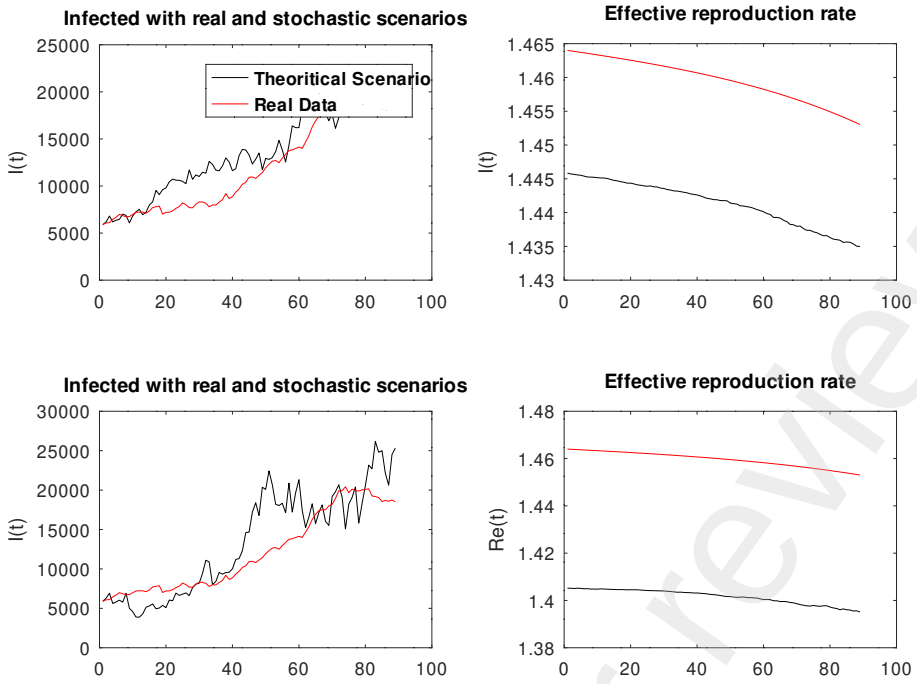

Figure 4: Trajectories of infected individuals in the stochastic epdemic model and confirmed COVID19 cases in Morocco.

is denoted by $\sigma_2 I(t)dW_2(t)$ and $\sigma_3 Q(t)dW_3(t)$. Moreover, it follows from historical fluctuations and data fitting of the epidemic model that the quarantined and slightly the infected size will remain in the same range of changes in the following 15 days. Besides, for more risk management and practical reasons, it is suitable to include greater values for the volatilities such as $\sigma_2 = 0.316$ and $\sigma_3 = 0.386$ for the bigger range illustrated in the figures (2) and (3). Furthermore, to avoid uncontrollable situation, decision makers must take in account a worse case scenario to not exceed the limit capacity of hospital beds by building fields hospitals. This strategy was adopted by Moroccan authorities by constructing temporary military field hospitals in Casablanca city, Ben Slimane and Ben Guerir military base and it helped to hospitalize and isolate the severe cases in a major wave time of infection. In the actual world, a population would hardly be completely vulnerable to an illness. Some people will be immune, for instance, as a consequence of a past illness that provided life-long immunity or prior vaccination. As a result, not all contacts will get affected, resulting in a reduced average number of confirmed cases per infected individuals than the basic reproduction number $\mathcal{R}_0$. Therefore, in figure (4), we illustrate the effective reproductive number $\mathcal{R}_e$ [30], which represents the average number of secondary cases per an existing infectious individual with a population containing vulnerable and immune individuals to the infectious disease and can be established from the product of a fraction of susceptible population and the basic reproductive number. Hence, we construct a random scenario for the infected compartment in the stochastic epidemic model (3) using Euler-Maruyama approximation [31] and we illustrate the real data of COVID19 cases in Morocco, where in the first row, we simulate a stochastic scenario with $\sigma_2 = 0.17$ and in the second row the volatility is defined as $\sigma_2 = 0.316$. In the second column, we illustrate the effective reproduction number for the deterministic and the stochastic cases. The technique provided in [30] for a compartmental model is centred on a deterministic characterization of the actual epidemic process. As result, the constructed trajectories of the effective reproductive number are deterministic. Therefore, we include the established stochastic quantity $\mathcal{R}_s$ instead of the

formal basic reproduction number to show the impact of a volatility on the trajectory of an effective reproduction number. Hence, a higher volatility is described in the second row (4), where there are large fluctuations without a clear trend can lead to a lesser effective reproduction number, which means that the more an environmental noise exists, the more it effects the spread of disease in time leading to a radical change in the behaviour of the dynamics.

4 Interpretation and discussion

In this work, a SIQRS stochastic epidemic model with logistic growth, general incidence function and non-pharmaceutical interventions is investigated. The corresponding stochastic epidemic model is studied according to an established stochastic threshold $\mathcal{R}_s$ with an addition of the impact of the logistic growth of a certain population. This allows to explore the extinction and the persistence of the infectious disease. In the last part of the investigation, we apply this model to COVID19 dataset in Morocco, where we fit the existed data to the model using least square method. Also, we illustrated a forecasting of 15 days according to the previous 64 days data. The presented investigation will contribute to explore stochastic epidemic systems. However, since the preventive measures can be switched depending of the circumstances. There are still more challenges for an extensive understanding of perturbed epidemic models such as including Markovian switching [32] and stochastic optimal control problems [33, 34] for the preventive measures.

References

[1] W. O. Kermack, A. G. McKendrick. Contributions to the mathematical theory of epidemics, Proc. Roy. Soc. A. 115 (772)(1927)700-721.

[2] B. Han, B. Zhou, D. Jiang, T. Hayat, A. Alsaedi. Stationary solution, extinction and density function for a high-dimensional stochastic SEI epidemic model with general distributed delay. Applied Mathematics and Computation Volume 405 (2021).

[3] B. Zhou, B. Han, D. Jiang, T. Hayat, A. Alsaedi. Ergodic stationary distribution and extinction of a hybrid stochastic SEQIHR epidemic model with media coverage, quarantine strategies and pre-existing immunity under discrete Markov switching. Applied Mathematics and Computation volume 410 (2021).

[4] Q. Liu, D. Jiang, T. Hayat, A. Alsaedi, B. Ahmad. Dynamical behavior of a higher order stochastically perturbed SIRI epidemic model with relapse and media coverage. Chaos, Solitons & Fractals Volume 139 (2020).

[5] M. El Fatini, R. Pettersson, I. Sekkak, T. Regragui. A stochastic analysis for a triple delayed SIQR epidemic model with vaccination and elimination strategies. J. Appl. Math. Comput. 64, 781-805 (2020).

[6] T. Caraballo, M. El Fatini, I. Sekkak, R. Taki, A. Laaribi. A stochastic threshold for an epidemic model with isolation and a non linear incidence. Communications on Pure & Applied Analysis, 2020, 19 (5) : 2513-2531.

[7] S. P. Rajasekar, M. Pitchaimani. Ergodic stationary distribution and extinction of a stochastic SIRS epidemic model with logistic growth and nonlinear incidence. Appl. Math. Comput. 377 (2020), 125143, 15 pp.

[8] S. P. Rajasekar, M. Pitchaimani, Q. Zhu. Progressive dynamics of a stochastic epidemic model with logistic growth and saturated treatment. Phys. A 538 (2020), 122649, 20 pp.

[9] Q. Liu, D. Jiang, T. Hayat, A. Alsaedi, B. Ahmad. A stochastic SIRS epidemic model with logistic growth and general nonlinear incidence rate. Phys. A 551 (2020), 124152, 11 pp.

[10] B. Han, D. Jiang, B. Zhou, T. Hayat, A. Alsaedi. Stationary distribution and probability density function of a stochastic SIRSI epidemic model with saturation incidence rate and logistic growth. Chaos Solitons Fractals 142 (2021), Paper No. 110519, 15 pp.

[11] J. Duhon, N. Bragazzi, J. D. Kong. The impact of non-pharmaceutical interventions, demographic, social, and climatic factors on the initial growth rate of COVID-19: A cross-country study. Science of the total environment 760 (2021) 144325.

[12] M. Betti, N. L. Bragazzi, J. M. Heffernan, J. D. Kong, A. Raad. Integrated vaccination and nonpharmaceutical interventions based strategies in Ontario, Canada, as a case study: a mathematical modelling study. J. R. Soc. Interface 18.

[13] S. Flaxman, S. Mishra, A.Gandy. Estimating the effects of non-pharmaceutical interventions on COVID-19 in Europe. Nature 584, 257-261 (2020).

[14] V. Capasso and G. Serio, A generalization of the Kermack-McKendrick deterministic epidemic model. Math. Biosci. 42(1978) 41-61.

[15] A.W.D. Edridge, J. Kaczorowska, A.C.R. Hoste, et al. Seasonal coronavirus protective immunity is short-lasting. Nat Med 26, 1691-1693 (2020).

[16] Z. Cao, W. Feng, X. Wen, ; L. Zu, M. Cheng. Dynamics of a stochastic SIQR epidemic model with standard incidence. Phys. A 527 (2019), 121180, 12 pp.

[17] G. Lan, Z. Chen, C. Wei, S. Zhang. Stationary distribution of a stochastic SIQR epidemic model with saturated incidence and degenerate diffusion. Phys. A 511 (2018), 61-77.

[18] Q. Liu, D. Jiang, T. Hayat, A. Alsaedi. Dynamics of a stochastic multigroup SIQR epidemic model with standard incidence rates. J. Franklin Inst. 356 (2019), no. 5, 2960-2993.

[19] M. El Fatini, M. El Khalifi, R. Gerlach, A. Laaribi, R. Taki. Stationary distribution and threshold dynamics of a stochastic SIRS model with a general incidence. Physica A Volume 534 (2019).

[20] A. Lahrouz, H. El Mahjour, A. Settati, A. Bernoussi, Dynamics and optimal control of a non-linear epidemic model with relapse and cure, Physica A 496 (2018), 299-317.

[21] A. Korobeinikov. Global Properties of Infectious Disease Models with Nonlinear Incidence. Bulletin of Mathematical Biology volume 69, pages 1871-1886 (2007).

[22] A. Lahrouz. Dynamics of a delayed epidemic model with varying immunity period and nonlinear transmission. International Journal of Biomathematics Vol. 08, No. 02, 1550027 (2015).

[23] Van den Driessche P, P., and J. Watmough. 2002. Reproduction numbers and sub-threshold endemic equilibria for compartments models of disease transmission. Mathematical biosciences 180: 29-48.

[24] X. Mao. Stochastic Differential Equations and Applications. Woodhead Publishing (2007).

[25] R.Z. Has'minskii, Stochastic Stability of Differential Equations, Sijthoff Noordhoff, Alphen aan den Rijn, The Netherlands, 1980.

[26] A. Hammoumi, R. Qesmi. Impact assessment of containment measure against COVID-19 spread in Morocco. Chaos, Solitons & Fractals Volume 140 (2020).

[27] M. El Fatini, M. El Khalifi, R. Gerlach, R. Pettersson. Bayesian forecast of the basic reproduction number during the Covid-19 epidemic in Morocco and Italy. Mathematical Population Studies (2021).

[28] H. Ferjouchia, A. Kouidere, O. Zakary, M. Rachik. Optimal control strategy of COVID-19 spread in Morocco using SEIRD model. Moroccan J. of Pure and Appl. Anal. Volume 7(1), 2021, Pages 66-79.

[29] A. Lahrouz, L. Omari, D. Kiouach. Global analysis of a deterministic and stochastic nonlinear SIRS epidemic model. Nonlinear Analysis: Modelling and Control, 2011, Vol. 16, No. 1, 59-76.

[30] A. Cintrón-Arias, C. Castillo-Chávez, L. M. A. Bettencourt, A. L. Lloyd, H. T. Banks. The estimation of the effective reproductive number from disease outbreak data. Mathematical Biosciences and Engineering, 2009, 6(2): 261-282.

[31] P.E. Kloeden, E. Platen. Numerical Solution of Stochastic Differential Equations. Springer (1992), Berlin.

[32] Q. Liu, D. Jiang, N. Shi. Threshold behavior in a stochastic SIQR epidemic model with standard incidence and regime switching. Appl. Math. Comput. 316 (2018), 310-325.

[33] H. Lu, Y. Ding, S. Gong, S. Wang. Mathematical modeling and dynamic analysis of SIQR model with delay for pandemic COVID-19. Mathematical Biosciences and Engineering 2021, Volume 18, Issue 4: 3197-3214.

[34] D. I Ketcheson. Optimal control of an SIR epidemic through finite-time non pharmaceutical intervention. J. Math. Biol. 83 (2021), no. 1, Paper No. 7, 21 pp.