Category Theory and Higher Dimensional Algebra: potential descriptive tools in neuroscience*

Ronald Brown Timothy Porter†

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Abstract

We explain the notion of colimit in category theory as a potential tool for describing structures and their communication, and the notion of higher dimensional algebra as potential yoga for dealing with processes and processes of processes.

Introduction

There does seem to be a problem in neuroscience in finding a language suitable for describing brain activity in a way which could lead to deduction, evaluation of theories, and even perhaps calculation. This is especially so when dealing with questions of global activity, as against activity of individual organs or cells. How can we bridge the gap between neuronal activity and what are variously described in the literature as percepts, concepts, thoughts, emotions, ideas, and so on? What are the relations between the meanings of these various words? When should we use one rather than another?

One hope is that a mathematics will arise which could help in these problems. To encourage such a mathematics, we need a dialogue between mathematicians and neuroscientists. Mathematicians can contribute by showing the way mathematics works, describing processes such as abstraction, concept refinement, etc., explaining what is currently available, and analysing the deficiences of current mathematics in helping with these problems. It is likely to be a difficult process to move towards such new mathematics, since life has evolved for a long period, whereas language and mathematics are relatively recent. Yet the ability to do mathematics is itself a result of evolution, and mathematics has a good track record in scientific discovery.

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†Mathematics Division, School of Informatics, University of Wales, Bangor, Gwynedd LL57 1UT, U.K..
email: {r.brown,t.porter}@bangor.ac.uk http://www.bangor.ac.uk/∼{mas010,mas013}

We cannot expect too much very quickly, since the mathematics so far developed may not be appropriate for this particular task. On the other hand it is potentially useful to see how mathematics deals with particular problems, such as the relations between words, and also to see what is now available which might have potential uses. Just as the physical sciences of physics and chemistry have strongly stimulated the development of areas of mathematics, so we can hope that the same will eventually hold for the neurological sciences.

1 Relations between words

The Greeks devised the axiomatic method, but thought of it in a different manner to that we do today. One can imagine that the way Euclid's Geometry evolved was simply through the delivering of a course covering the established facts of the time. In delivering such a course, it is natural to formalise the starting points, and so arranging a sensible structure. These starting points came to be called postulates, definitions and axioms, and they were thought to deal with real, or even ideal, objects, named points, lines, distance and so on. The modern view, initiated by the discovery of non Euclidean geometry, is that the words points, lines, etc. should be taken as undefined terms, and that axioms give the relations between these. This allows the axioms to apply to many other instances, and has led to the power of modern geometry and algebra.

This suggests a task for the professionals in neuroscience, in order to help a mathematician struggling with the literature, namely to devise some kind of glossary with clear relations between these various words and their usages, in order to see what kind of axiomatic system is needed to describe their relationships. Clarifying, for instance, the meaning to be ascribed to 'concept', 'percept', 'thought', 'emotion', etc., and above all the relations between these words, is clearly a fundamental but difficult step.

2 Category theory and colimits: gluing and structure

One of the strong developments in mathematics of the 20th century has been that of category theory, with its power of describing the processes of mathematics, developing new logics, unifying different topics, and revealing underlying abstract processes which have turned out to have wide implications and uses.

Abstraction allows analogies, by encoding relations, and relations among relations. As an example, when we note that $2+3=3+2$ and $2\times 3=3\times 2$, and extend this to the abstract *commutative law* $x\circ y=y\circ x$ for a binary operation $\circ$, we are making an analogy between addition and multiplication, and also make this law available in other situations. One may presume that the power of abstraction, in some sense of making maps, must be deeply encoded in evolutionary history as a technique for encouraging survival, since a map gives a small and manipulable model of the environment. Symbolic manipulation in mathematics often involves *rewriting*, an example of which is using the commutative law, i.e. replacing in a complicated formulae various instances of $x\circ y$ by $y\circ x$. We do this rewriting for example in obtaining the equation

$$3\times 2\times 5\times 3\times 2\times 5\times 3=2^2\times 3^3\times 5^2.$$

For more on rewriting, see [4].

A study of why and how mathematics works could be useful for making models for neurological functions involving maps of the environment. Mathematics may also possibly provide a comprehensible case study of the evolution of complex interacting structures, and so may yield analogies helpful for developing and evaluating models of brain activity, in order to derive better models, and so better understanding. We expect to need a new language, a new mathematics, for describing brain activity. To see what is involved in this search, it is reasonable to study the evolution of mathematics, and of particular branches such as category theory.

Some description of category theory is given in [10]. The relevance to biological development is described in a series of papers by Ehresmann and Vanbremersch [15]. In particular, they see the notion of *colimit* in a category as describing a structure made up of inter-related parts, so that a category evolving with time can then allow evolving structures, the structures being given as colimits in the category C_t at time t.

This notion of colimit gives a very general setting in which to describe the process of *gluing* or *amalgamating* complex structures, together with a description of the method of input to and output from these structures.

First we need to give the notion of category. This developed from a useful notation for a function: moving from the somewhat obscure 'a function is $y = f(x)$ where y varies as x varies' to the clearer 'a function $f : X \to Y$ assigns to each element x of the set X an element $f(x)$ of the set Y'. This sees 'function' as being a 'process'. The composition of functions then suggests the first step in the notion of a **category** C, which consists of a class $Ob(\mathsf{C})$ of 'objects', a set of 'arrows', or 'morphisms' $f : X \to Y$ for any two objects X, Y, and a composition, giving, for instance, $fg : X \to Z$ if also $g : Y \to Z$. This composite is represented by the diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \tag{1}$$

or even by

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "fg"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd} \tag{2}$$

A category thus has not only a composition structure but also a 'position' structure given by its class of objects. The only rules are associativity $f(gh) = (fg)h$ when both sides are defined, and the existence of identities 1_X at each object X, so that with f as above, $1_X f = f = f 1_Y$.

The notion of colimit in a category generalises the notion of forming the union $X \cup Y$ of two overlapping sets, with intersection $X \cap Y$. However, instead of concentrating on the sets X, Y themselves, we place them in context, and say that the utility of the union is that it allows us to construct *functions* $f : X \cup Y \to C$ for any C by specifying functions $f_X : X \to C, f_Y : Y \to C$ which agree on the intersection $W = X \cap Y$. So we replace the specific construction of $X \cup Y$ by a property which describes, using functions, the relation of this construction to all other sets. That is, the emphasis is on the relation between input and output.

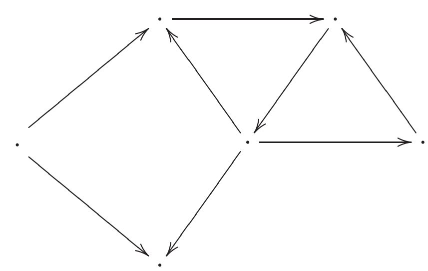
A colimit has 'input data', a 'cocone'. In the case of $X \cup Y$, this cocone consist of the two functions $i_X : X \cap Y \to X,\ i_X : X \cap Y \to Y$.

There are similar situations in other contexts. The numbers

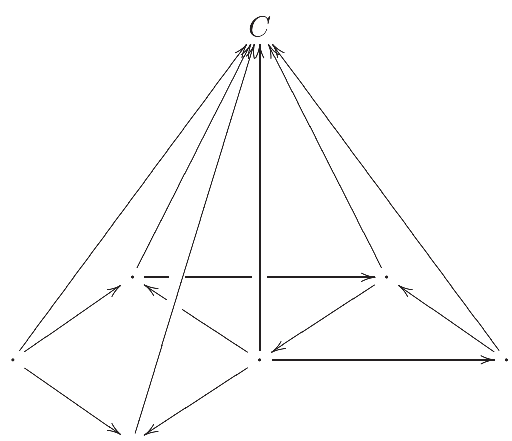
$$\max(a, b), \text{ and } \mathrm{lcm}(a, b)$$

are all constructed from their 'parts', the numbers a, b. Here the 'arrows' of our previous 'sets and functions' example are replaced in the first case by the order relation '$x \leqslant y$', 'x is less than or equal to y', and in the second case by the divisibility relation '$x|y$', 'x divides y'. So we have the rules that: (i) if $a \leqslant c$ and $b \leqslant c$ then $\max(a, b) \leqslant c$, (ii) if $a|c$ and $b|c$ then $\mathrm{lcm}(a, b)|c$. Thus to make analogies between constructions for many different mathematical structures, we simply formulate a notion in a general category – that is all! In this way category theory has been a great unifying force in mathematics of the 20th century, and continues so to do.

We also generalise to more complex input data. So the 'input data' for a colimit is a *diagram* D, that is a collection of some objects in a category C and some arrows between them, such as:

$D =$ 

Next we need 'functional controls': this is a *cocone with base D and vertex an object C.*



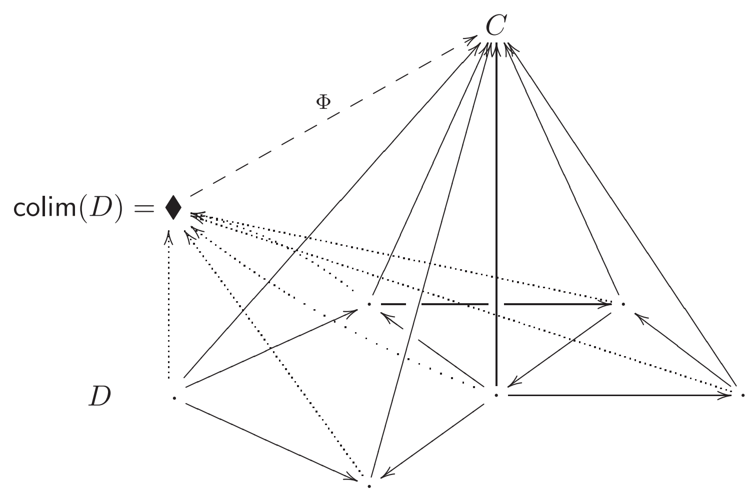
such that each of the triangular faces of this cocone is commutative.

The output from such input data will be an object $\mathsf{colim}(D)$ in our category C defined by a special *colimit cocone* such that any cocone on D factors *uniquely* through the colimit cocone. The commutativity condition on the cocone in essence forces, in the colimit, an interaction of the images of different parts of the diagram D. The uniqueness condition makes the colimit the *best possible* solution to this factorisation problem

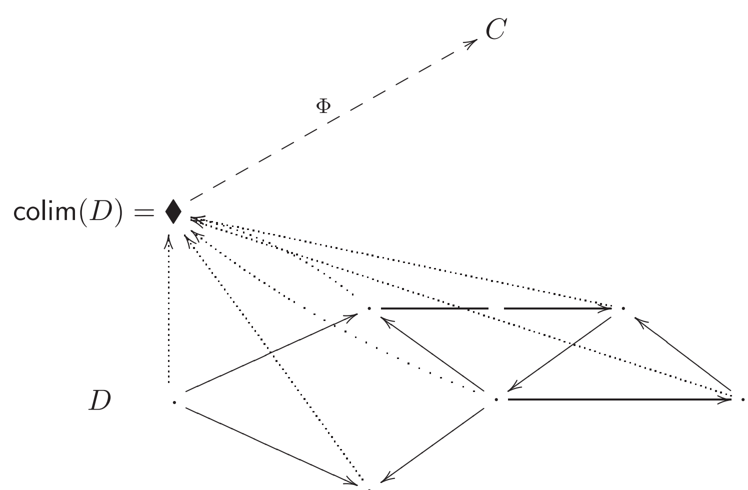
In the next picture the colimit is written

$$\blacklozenge = \mathsf{colim}(D),$$

the dotted arrows represent new morphisms which combine to make the colimit cocone:



and the broken arrow Φ is constructed from the other information. Again, all triangular faces of the combined picture are commutative. Now stripping away the 'old' cocone gives the factorisation of the cocone via the colimit:



Intuitions:

The object colim(D) is 'put together' from, or 'composed of the parts of', the constituent diagram D by means of the colimit cocone. From beyond (or above our diagrams) D, an object C 'sees' the diagram D 'mediated' through its colimit, i.e. if C tries to interact with the whole of D, it has to do so via colim(D). The colimit cocone can be thought of as a kind of program: given any cocone on D with vertex C, the output will be a morphism

$$\Phi : \mathsf{colim}(D) \to C$$

constructed from the other data. How is this morphism realised, what are its values?

To focus on a common example, consider the process of sending an email document, call it E. To send this we need a server S, which breaks down the document E into many parts E_i for i in some indexing set I, and labels each part E_i so that it becomes E_i'. The labelled parts E_i' are then sent to various servers S_i which then send these as messages E_i'' to a server S_C for the receiver C. The server S_C combines the E_i'' to produce the received message M_E at C. Notice also that there is an arbitrariness in breaking the message down, and in how to route through the servers S_i, but the system is designed so that the received message M_E is independent of all the choices that have been made at each stage of the process. A description of the email system as a colimit may be difficult to realise precisely, but this analogy does suggest the emphasis on the amalgamation of many individual parts to give a working whole, which yields exact final output from initial input, despite choices at intermediate stages.

One question for neuroscientists is: does the brain use analogous processes for communication between its various structures? What we can say is that this general colimit notion represents a general mathematical process which is of fundamental importance in describing and calculating with many algebraic and other structures.

To go back to our email analogy, the morphism Φ is constructed on some element z of colim(D), by splitting z up somehow into pieces z_i which come from parts z_i'' in objects of D, mapping these z_i'' over using the cocone on D, and combining them in C. This is how some proofs that certain cocones are colimits are actually carried out, see [7].

Thus a conjecture as far as biological processes are concerned, is that this notion of colimit may give useful analogies to the way complex systems operate. More generally, it seems possible that this particular concept in category theory, seeing how a big object is built up of smaller related pieces, may be useful for the mathematics of processes.

3 Higher dimensional algebra

The aim here is to explain some mathematical ideas with which the authors have been preoccupied since the 1960s and 1970s respectively. There was a lot of experimentation to produce the mathematics which would encompass some apparently simple intuitions. This experimentation can be viewed as 'extraction of concepts', and it also exemplifies mathematical concepts that might provide a model of inter-relationships that is much freer than the usual ones. Indeed, higher dimensional algebra is being used to model distributed systems.

A basic idea is that we may need to get away from 'linear' thinking in order to express intuitions clearly.

Thus the equation

$$2 \times (5 + 3) = 2 \times 5 + 2 \times 3 \tag{3}$$

is more clearly shown by the figure

$$\begin{array}{ccc} |\,|\,|\,|\,| & \quad & |\,|\,| \\ |\,|\,|\,|\,| & \quad & |\,|\,| \end{array} \tag{4}$$

Indeed the number of conventions you need to understand equation (3) make it seem barbaric compared with the picture (4). It is also interesting to see how you could express in a picture the general linear formula of the distributivity law

$$a \times (b + c) = a \times b + a \times c.$$

The importance of having simple comprehensible pictures instead of complex formulae is that the pictures help one to imagine theorems and their proofs. (The above exposition is borrowed from an account in week 53 of John Baez's series 'This week's find in mathematical physics' [6].)

Those who have read Edwin Abbott's famous book 'Flatland' [1] (and those who have not, have a delight in store!) will recall the limited interactions available to the inhabitants of Lineland. It seems unreasonable to suppose that a purely linear mathematics can express reasonably the complex interactions that occur in the brain.

We often translate geometry into algebra. For example, a figure as follows:

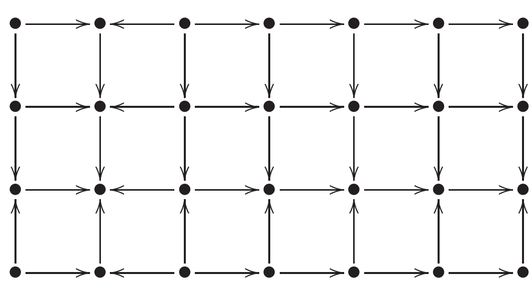
$$\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \xrightarrow{c} \bullet \xrightarrow{d} \bullet$$

is easily translated into

$$abcd$$

and the language for expressing this is again that of *category theory*. It is useful to express this intuition as 'composition is an algebraic inverse to subdivision'. The labelled subdivided line gives the composite word, $abcd$.

But how do we express a diagram such as the following



where the squares are supposed filled and labelled? It seems that in doubling the number of dimensions from 1 to 2, you need to move from categories to *double categories*, or something similar, based on directed squares rather than on arrows.

The extra richness involved is that a square can have more complicated relations to other squares than can happen in the linear situation.

Such questions arose from a *gluing* or colimit problem in topology, namely to describe the behaviour of a big object in terms of the behaviour of its parts. When the first author started work on this, the particular problem in dimension 1 was well known and solved, but the question was to carry out similar methods in higher dimensions. For a survey of this for a mathematical audience, see [7].

In the topological situation from which these ideas arose, to obtain the uniqueness in the output of a colimit, as we always require for our emails, we had to go further than *'algebraic inverses to subdivision'* and to use also the notion of *'commutative cube'*.

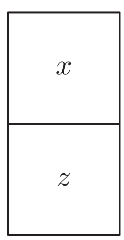
A commutative square is very easy:

$$\begin{tikzcd} \bullet \arrow[r, "a"] \arrow[d, "c"'] & \bullet \arrow[d, "b"] \\ \bullet \arrow[r, "d"'] & \bullet \end{tikzcd}$$

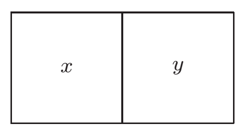
is easily translated into mathematics as

$$ab = cd, \qquad \text{or} \qquad a = cdb^{-1}. \tag{5}$$

The surprising thing is that to determine a *commutative cube* needs some new ideas. First we need to know how to compose the square faces of a cube. This can be done in two directions:



$x \circ_1 z$



$x \circ_2 y$

So we get a notion of a **double category**, whose elements are squares, for which in the above diagram the composition $x \circ_1 z$ is defined if and only if the bottom edge of x is the same as the top edge of z, and similarly (but right and left edges) for $x \circ_2 y$. Thus the compositions are partially defined, under geometric conditions. This enables a close relation between the algebra and geometry. On these compositions, we have to impose all the obvious geometric rules.

These rules enables an easy description of multiple compositions. But there remains the question of how to define a *commutative cube*? A cube has six faces, which can divide into two groups of three. With clear conventions, one would like to equate the two compositions

$$\begin{array}{|c|c|}\hline \partial_2^0 & \partial_3^1 \\ \hline \partial_1^1 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \qquad\qquad \begin{array}{c|c|}\cline{2-2} & \partial_1^0 \\ \hline \multicolumn{1}{|c|}{\partial_3^0} & \partial_2^1 \\ \hline \end{array} \tag{6}$$

Unfortunately, this does not make sense because the little squares do not form a rectangular array, and also because the edges of each block are not correct to give equality. It is interesting that the extension from squares in dimension 2 to cubes in dimension 3 produces these gaps which require a new set of concepts to handle them.

We need new elements to fill in the corners, and in fact you also need to expand out, to obtain the equality of:

$$\begin{array}{|c|c|c|}\hline = & \partial_2^0 & \partial_3^1 \\ \hline \ulcorner & \partial_1^1 & \lrcorner \\ \hline \end{array} \qquad\qquad \begin{array}{|c|c|c|}\hline \ulcorner & \partial_1^0 & \lrcorner \\ \hline \partial_3^0 & \partial_2^1 & = \\ \hline \end{array} \tag{7}$$

Thus 2-dimensional algebra needs some new basic constructions. This is not surprising. In dimension 1, you are limited to staying still, moving forward, or moving backward. In dimension 2, you can also turn left or right. This is what needs to be modelled formally.

We need some special squares called *thin squares*:

$$\begin{pmatrix} 1 & \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} & 1 \end{pmatrix} \quad \begin{pmatrix} a & \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} & a \end{pmatrix} \quad \begin{pmatrix} 1 & \begin{smallmatrix} b \\ b \end{smallmatrix} & 1 \end{pmatrix}$$

$$\square \qquad = \text{ or } \varepsilon_2 a \qquad |\,| \text{ or } \varepsilon_1 b$$

of which the above first three are forms of horizontal and vertical identities, which correspond to not moving in certain directions, while our last two

$$\begin{pmatrix} a & \begin{smallmatrix} a \\ 1 \end{smallmatrix} & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & \begin{smallmatrix} 1 \\ a \end{smallmatrix} & a \end{pmatrix}$$

$$\lrcorner \text{ or } \Gamma a \qquad \ulcorner \text{ or } \Gamma' a$$

are known as *connections*, and correspond to turning left or right. How do these all interact?

The rules on the connections are as follows:

(i) $[\ulcorner\ \lrcorner] = |\ |, \quad [\,\Gamma' a\ \Gamma a\,] = \varepsilon_1 a;$

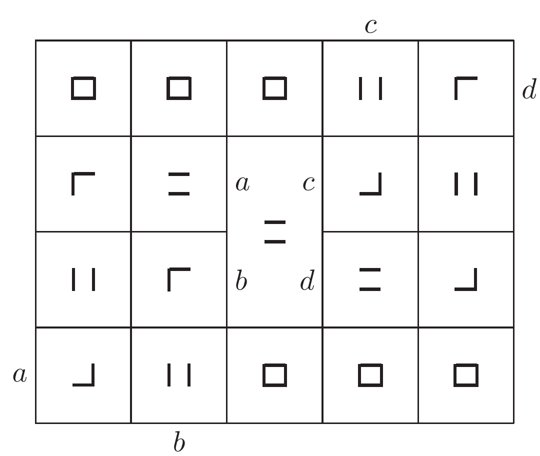
(ii) $\begin{bmatrix}\ulcorner\\ \lrcorner\end{bmatrix} = \text{=}\,, \quad \begin{bmatrix}\Gamma' a\\ \Gamma a\end{bmatrix} = \varepsilon_2 a;$

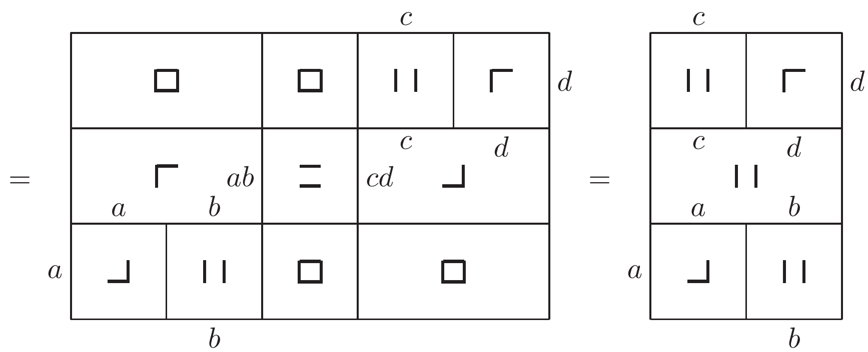
(iii) $\begin{bmatrix}\ulcorner & \text{=}\\ |\ | & \ulcorner\end{bmatrix} = \ulcorner, \quad \begin{bmatrix}\Gamma' a & \varepsilon_2 a\\ \varepsilon_1 a & \Gamma' b\end{bmatrix} = \Gamma'(ab);$

(iv) $\begin{bmatrix}\lrcorner & |\ |\\ \text{=} & \lrcorner\end{bmatrix} = \lrcorner, \quad \begin{bmatrix}\Gamma a & \varepsilon_1 b\\ \varepsilon_2 b & \Gamma b\end{bmatrix} = \Gamma(ab).$

Notice that we give two notations for these thin squares, and that the more heiroglyphic notation is much easier to grasp by the eye. This emphasises the idea that the history of mathematics is much involved with the history of improved notation. The first two rules for connections can be thought of as formalising turning left and then right (or the other way round), while the last two formalise that turning with an arm outstretched is the same as just turning.

Here we just have time to show a typical calculation to convince you that 2-dimensional rewriting looks like a new kind of manipulation. We start with the first complex diagram and use the rules to rewrite it to a simpler one. The rewriting may also be done a different way to give a different expression which has, by the rules, an equal evaluation in the double category with connections.





Analogous rewriting has been carried out in three dimensions in [2], and, as may be imagined, is not easy to handle. As always, the development of a new mathematics solves some problems and then brings a range of new problems into view.

Rewriting, changing one formula to another according to certain rules, is a basic facet of much mathematics. When multiple rewrites are occurring, for various operations, the study of this ‘distributed’ or ‘concurrent’ rewriting can often be pictured as involving higher dimensional laws. Models of the complex message passing and distributed rewriting in the brain have yet to find the laws relevant to that context. Such laws could be key elements for the next level of understanding of cognition.

Questions

Is the colimit notion useful to describe the way the brain (or a brain module) integrates structural information incoming in various forms to give a determined output?

What other models are there? It is good to start with the assumption that the simplest idea works!

Information is often ‘subdivided’ by the sensory organs and is reintegrated by the brain. To enable different parts of that information to be integrated, there must be some ‘glue’, some inter-relational information available. If we are given arrows a, b, c, d with no information on where they start or end, then we could form combinations which make no geometric sense. The colimit/composition process makes sense only where the inter-relations are also such as to enable the ‘integration’ to be well defined. Higher dimensional algebra allows more complex notions of ‘well formed composition’, and ones more adapted to geometry.

Computation and computer science

A mathematician always wants to do sums, get explicit answers to some situations, but recognises that not every sum can be worked out. In fact you may need more mathematics to work out how to do the sums.

Computer languages do not yet seem to be good at doing *structural mathematics*, or even at expressing a high level algorithm, i.e. one which works at a high structural level. Indeed, current computer languages were not designed with the needs of mathematics in mind. Yet algorithms designed to encode a high level of structural information should be more efficient – we do not tell people the way to the station by giving information on all the cracks in the pavement. We need to map the landscape, that is give a usable model of it, to describe the path through it.

Since evolution is ‘concerned with’ efficiency, we must expect that the brain has evolved methods for dealing with structural information. It looks like a reasonable conjecture that category theory and higher dimensional category theory could be necessary for modelling this kind of behaviour.

Higher Dimensional Algebra has already shown its use in models of information management, and in concurrency. Descriptions of systems by graphs are well known, with development described algebraically by paths in graphs. Interacting systems need higher dimensional graphs, and a generalisation of the notion of path. Higher Dimensional Algebra is still young, and there are many new possibilities opened up, as a web search shows.

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